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ALGEBRAIC TREATMENT OF A FINITE-DIFFERENCE EQUATION FOR A N -DIMENSIONAL RELATIVISTIC COULOMB PROBLEM

S.H.M. NAGIYEV

Institute of Physics, Academy of Sciences of Azerbaijan

H. Gavid str. 33, Baku 370143

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An algebraic solution of a Coulomb problem in the N -dimensional relativistic configuration \vec{r}_s -space is presented.

1. Introduction

In the papers [1-6] in the framework of quasipotential approach [7,8] a version of the relativistic quantum mechanics has been developed. It is based on the concept of the relativistic 3-dimensional configuration \vec{r} -space [1], which arises when we use the Fourier expansion over the relativistic

"plane waves" [9] $\left(\frac{\hbar}{2\pi} = m = c = 1\right)$

$$\begin{aligned} \langle \vec{r} | \vec{p} \rangle &= (p_0 - \vec{p} \vec{n})^{-1-i\epsilon} \\ \vec{r} &= r \vec{n}, \quad 0 < r < \infty, \quad \vec{n}^2 = 1 \end{aligned} \quad (1.1)$$

instead of the usual plane waves $e^{i\vec{p}\vec{r}}$. The system of functions (1.1) is a complete and orthogonal one on the mass hyperboloid $p_0^2 - \vec{p}^2 = 1$ of a unit mass particle. From the geometrical point of view, on the mass hyperboloid $p_0^2 - \vec{p}^2 = 1$ the 3-dimensional Lobachevsky space is realized, the group of motions of which is the Lorentz group $SO(3,1)$.

In the nonrelativistic limit

$$\lim_{c \rightarrow \infty} \langle \vec{r} | \vec{p} \rangle = e^{i\vec{p}\vec{r}} \quad (1.2)$$

The relativistic quantum mechanics in \vec{r} -space carries many important features of quantum mechanics, but in distinction from quantum mechanics the equation for the wavefunction of the relativistic motion is written in a

finite-difference form. Thus the relativistic generalizations of some exactly solvable problems of quantum mechanics (harmonic oscillator, square-well and Coulomb problem) has been considered [2-6].

There are also the algebraic approaches in the description of quantum systems by the theory of dynamical symmetries [10,11], potential groups [12], Casimir operators with mixed spectrum [13].

The purpose of the present paper is to solve a finite-difference equation that describes a Coulomb problem in a N -dimensional relativistic configuration space by Lie algebraical method.

2. N -dimensional relativistic configuration \vec{x}_N -space

The N -dimensional relativistic configuration \vec{x}_N -space is introduced [14] by analogy with the 3-dimensional relativistic configuration \vec{r} -space. Namely, transition to the \vec{x}_N -space

$$\psi(\vec{x}_N) = \frac{1}{(2\pi)^{N/2}} \int \frac{d\vec{p}_N}{p_0} \langle \vec{x}_N | \vec{p}_N \rangle \psi(\vec{p}_N) \quad (2.1)$$

is performed by the expansion in terms of the matrix elements

$$\begin{aligned} \langle \vec{x}_N | \vec{p}_N \rangle &= (p_0 - \vec{p}_N \cdot \vec{n}_N)^{\frac{N-1}{2} - i r} \\ \vec{x}_N &= r \vec{n}_N, \quad \vec{n}_N^2 = 1 \end{aligned} \quad (2.2)$$

of the infinite - dimensional unitary representations of the N -dimensional momentum Lobachevsky space motion group $SO(N,1)$, which is realized on the upper sheet of the hyperboloid $p_0^2 - \vec{p}_N^2 = \equiv p_0^2 - p_1^2 - p_2^2 - \dots - p_N^2 = 1$.

The N -dimensional relativistic "plane waves" (2.2) are the eigenfunctions of the free Hamiltonian $H_0^{(N)}$, which in the spherical system of coordinates $\vec{x}_N = (r, \theta_1, \theta_2, \dots, \theta_{N-1})$ has the form

$$H_0^{(N)} = c \hbar i \partial_r + \frac{\hbar^2 (N-1)}{2r} s \hbar i \partial_s - \frac{\Delta_0}{r[2r - i(N-3)]} e^{i\theta} \quad (2.3)$$

where Δ_0 is the Laplace operator on the unit sphere S^{N-1} . The eigenfunctions of Δ_0 are the N -dimensional spherical harmonics $Y_{N1}(\theta_1, \theta_2, \dots, \theta_{N-1})$:

$$\Delta_0 Y_{N1}(\theta_1, \theta_2, \dots, \theta_{N-1}) = -l(l+N-2) Y_{N1}(\theta_1, \theta_2, \dots, \theta_{N-1}), \quad (2.4)$$

3. Coulomb problem in the N -dimensional relativistic configuration \vec{x}_N -space

Here we consider a simple generalization of the 3-dimensional relativistic Coulomb problem to N -dimensional case. In the N -dimensional relativistic configuration \vec{x}_N -space a Coulomb problem is described by the finite-difference equation

$$(H_0^{(N)} - \alpha/r)\psi(\vec{x}_N) = E\psi(\vec{x}_N) \quad (3.1)$$

Due to the geometrical $O(N)$ symmetry the angular dependence of the wavefunction is defined by the spherical harmonics

$$\psi(\vec{x}_N) = \left[(-r)^{\binom{N-1}{2}} \right]^{-1} \psi_{N1}(r) Y_{N1}(\theta_1, \theta_2, \dots, \theta_{N-1}) \quad (3.2)$$

Therefore, a problem is reduced to finding the eigenvalues and eigenfunctions of the radial part of a Hamiltonian

$$(H_0^{(N)rad} - \alpha/r)\psi_{N1}(r) = E\psi_{N1}(r) \quad (3.3)$$

where

$$H_0^{(N)rad} = ch i \partial_r + \frac{L(L+1)}{2r^{(2)}} e^{i\theta_r} \quad , \quad L = l + \frac{N-3}{2} \quad (3.4)$$

and $x^{(l)}$ is the generalized degree $x^{(l)} = i^l \frac{\Gamma(-ix + l)}{\Gamma(-ix)}$. To solve the equation (3.1) algebraically, we introduce three operators

$$\begin{aligned} \Gamma_0 &= rch i \partial_r + \frac{i(N-1)}{2} sh i \partial_r - \frac{\Delta_0}{2r-i(N-3)} e^{i\theta_r} \quad , \\ \Gamma_4 &= r \quad , \\ T &= rsh i \partial_r + \frac{i(N-1)}{2} ch i \partial_r - \frac{\Delta_0}{2r-i(N-3)} e^{i\theta_r} \end{aligned} \quad (3.5)$$

They satisfy the commutation relations of Lie algebra $SO(2,1)$:

$$[\Gamma_0, \Gamma_4] = iT \quad , \quad [\Gamma_4, T] = -i\Gamma_0 \quad , \quad [T, \Gamma_0] = i\Gamma_4 \quad (3.6)$$

The direct calculation shows that the Casimir operator is equal to

$$C_2 = \Gamma_0^2 - \Gamma_4^2 - T^2 = -\Delta_0 + \frac{(N-1)(N-3)}{4} = L(L+1), \quad (3.7)$$

From (3.7) it follows that to each value of the orbital quantum number corresponds an irreducible unitary representation $D+(-L-1)$ of the group $SO(2,1)$ with the eigenvalues of the compact generator Γ_0 , which are bounded below and are equal to $n = n_r + L + 1$, where $n_r = 0, 1, 2, \dots$ is the radial quantum number.

In terms of (3.5) the equation (3.1) (after multiplying by r) has the following form

$$(\Gamma_0 - E\Gamma_4 - \alpha)\psi = 0. \quad (3.8a)$$

By means of the unitary transformation $\tilde{\psi} = e^{-i\Theta r}\psi$, the eigenvalue equation (3.8a) can be written as

$$(a\Gamma_0 + b\Gamma_4 - \alpha)\tilde{\psi} = 0, \quad (3.8b)$$

where $a = ch\Theta - esh\Theta$, $b = sh\Theta - Ech\Theta$.

Let us consider the cases $|E| < 1$ and $|E| > 1$ in (3.8b) separately.

If $|E| < 1$, then the compact generator Γ_0 can be diagonalized.

Choosing the angle Θ in (3.8b) as $\Theta = \frac{1}{2} \ln \frac{1-E}{1+E}$, we reduce (3.8b) to the form

$$(\sqrt{1-E^2}\Gamma_0 - \alpha)\tilde{\psi} = 0 \quad (3.9)$$

From (3.9) we obtain a discrete energy spectrum for the N -dimensional relativistic Coulomb problem

$$E_n = \pm \sqrt{1 - \frac{\alpha^2}{n^2}}, \quad n = n_r + l + \frac{N-1}{2} \quad (3.10)$$

If $|E| > 1$, then the noncompact generator Γ_4 can be diagonalized. For this purpose choose $\Theta = \frac{1}{2} \ln \frac{1-E}{1+E}$ and obtain instead of (3.8b) the equation

$$[\sqrt{E^2 - 1}(\operatorname{sgn} E)\Gamma_4 + \alpha]\tilde{\psi} = 0 \quad (3.11)$$

Since Γ_c has the continuous real spectrum $\lambda \in R$, from (3.11) we find the continuous spectrum of the system

$$E_\lambda = \pm \sqrt{1 + \alpha^2 / \lambda^2} \quad (3.12)$$

Now we can also define the explicit form of the radial wavefunctions algebraically. They are expressed in terms of hypergeometric function $F(a, b; c; z)$.

In the case of the discrete spectrum, $E_n = \cos \chi_n$, we have

$$\psi_{\text{dis}}(r) = (-r)^{L+1/2} e^{-ix} F(-n, -ir + L + 1; 2L + 2; 1 - e^{-2ix})$$

and in the case of the continuous spectrum, $E_n = \cos \chi_n$, we have

$$\psi_{\text{cont}}(r) = (-r)^{L+1/2} e^{ix} F(L + 1, \frac{i\alpha}{\text{sh}\chi}, -ir + L + 1; 2L + 2; 1 - e^{-2ix})$$

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Ş.M. Nağıyev

N-ÖLÇÜLÜ RELYATİVİSTİK KULON MƏSƏLƏSİ ÜÇÜN SONLU-FƏRQ TƏNLİYİNƏ CƏBRİ YANAŞMA

N-ölçülü relyativistik \vec{r}_n -konfigurasiya fəzasında Kulon məsələsinin cəbri həlli verilmişdir.

Ш.М. Нагнов

АЛГЕБРАИЧЕСКИЙ ПОДХОД К КОНЕЧНО-РАЗНОСТНОМУ УРАВНЕНИЮ ДЛЯ N -МЕРНОЙ РЕЛЯТИВИСТСКОЙ КУЛОНОВСКОЙ ЗАДАЧИ

Дано алгебраическое решение кулоновской задачи в N -мерном релятивистском конфигурационном \vec{x}_N -пространстве.