MATHEMATICA SOFTWARE IN INTEGRATION OF CHIRAL FIELD MODEL

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The *Mathematica* algorithm for recurrent procedure of integration of the principal chiral field problem is presented.

1. *Mathematica* is the world's only fully integrated environment for technical computing. First released in 1988, it has had a profound effect on the way computers are used in many technical and other fields. In this article we show the application of this powerful tool to solution of the principal chiral field problem.

The principal chiral field problem may be written in the form:

$$g_\xi g^{-1} = f_\xi \ , \ g_\eta g^{-1} = -f_\eta \qquad (1),$$

where g and f are respectively, group and algebra-valued functions,
which are solutions of the principal chiral field problem

$$\left(g_\xi g^{-1}\right)_\eta + \left(g_\eta g^{-1}\right)_\xi = 0\ , \quad 2 f_{\xi\eta} = \left[f_\xi , f_\eta\right] \qquad (2)$$

In the case of $SL(2,C)$ algebra these equations can be solved completely for the following initial condition [1]:

$$f = \begin{pmatrix} \tau & \alpha \\ 0 & -\tau \end{pmatrix}\ , \ \tau_{\xi\eta} = 0\ , \ \alpha_{\xi\eta} = \alpha_\eta - \alpha_\xi \qquad (3)$$

The result of integration of the system (3) can be expressed in terms of chains of solutions of the following system of linear equations

$$\begin{aligned} \alpha[n+1]_\xi &= \alpha[n]_\xi - 2\alpha[n] \\ -\alpha[n+1]_\eta &= \alpha[n]_\eta + 2\alpha[n] \end{aligned} \qquad (4)$$

which are nothing more than Backlund transformations.
The solutions of (4) in the explicit form are presented in [2].
The discrete symmetry transformation allows carrying out the recurrent procedure of finding the solution of (2)

$$g_{n+1} = S_n w g_n$$

The solutions are expressed in terms of chains (4) starting from the 0-step (3).
We are using the following parameterization of the group element:

$$g_{n+1} = Exp(\alpha[n]X^+)Exp(t[n]h)Exp(\beta[n]X^-)$$

2. Below one can find the *Mathematica* program of the first two steps of recurrent procedure.
The *input* is written in Bold style, the results – in Normal.

In[1]:= $\mathbf{X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};\quad X^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};\quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};}$

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G0 = MatrixExp[n X+].MatrixExp[r h];
S1 = MatrixExp[s[1] X+].MatrixExp[s0[1] h]; G1 = S1.w.G0;
G1
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Out[4]= $\{\{-e^{r-s_0[1]}\,s[1],\ e^{-r+s_0[1]} - e^{-r-s_0[1]}\,n\,s[1]\},\ \{-e^{r-s_0[1]},\ -e^{-r-s_0[1]}\,n\}\}$

In[5]:= **G1 = FullSimplify[G1 /. s[1] → a[1] /. n → a[-1] /. s0[1] → Log[a[0]]]**

Out[5]= $\left\{\left\{-\frac{e^{r}\,a[1]}{a[0]},\ \frac{e^{-r}\,(a[0]^2 - a[-1]\,a[1])}{a[0]}\right\},\ \left\{-\frac{e^{r}}{a[0]},\ -\frac{e^{-r}\,a[-1]}{a[0]}\right\}\right\}$

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In[6]:= β[1] = G1[[2, 1]] / G1[[2, 2]]; α[1] = G1[[1, 2]] / G1[[2, 2]];
        t[1] = -Log[G1[[2, 2]]];
        FullSimplify[α[1]]
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Out[8]= $-\frac{a[0]^2}{a[-1]} + a[1]$

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In[9]:= FullSimplify[t[1]]

Out[9]= -Log[-(ⅇ^-τ a[-1])/a[0]]

In[10]:= β[1] = G1[[2, 1]] / G1[[2, 2]]

Out[10]= ⅇ^(2 τ)/a[-1]

In[11]:= s0[2] = Log[Det[{{a[0], a[1]}, {a[1], a[2]}}]/a[0]];

         s[2] = (a[0] Det[{{a[0], a[2]}, {a[1], a[3]}}] - a[1] Det[{{a[0], a[1]}, {a[1], a[2]}}])/a[0]^2;

         S2 = MatrixExp[s[2] X+].MatrixExp[s0[2] h];
         G2 = S2.w.G1;
         β[2] = G2[[2, 1]] / G2[[2, 2]];
         α[2] = G2[[1, 2]] / G2[[2, 2]];
         t[2] = -Log[G2[[2, 2]]];
         FullSimplify[β[2]]
         Null

Out[18]= (ⅇ^(2 τ) a[1])/(-a[0]^2 + a[-1] a[1])

In[20]:= FullSimplify[t[2]]

Out[20]= -Log[(ⅇ^-τ (a[0]^2 - a[-1] a[1]))/(a[1]^2 - a[0] a[2])]

In[21]:= FullSimplify[α[2]]

Out[21]= (a[1]^3 - 2 a[0] a[1] a[2] + a[-1] a[2]^2)/(a[0]^2 - a[-1] a[1]) + a[3]
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3. The last expression is nothing more than

$$\alpha[2]=\frac{Det\begin{pmatrix} a[-1] & a[0] & a[1]\\ a[0] & a[1] & a[2]\\ a[1] & a[2] & a[3]\end{pmatrix}}{Det\begin{pmatrix} a[-1] & a[0]\\ a[0] & a[1]\end{pmatrix}}$$

Using the expressions for the group-value elements and the relations (1), one can easily come to final expressions for the algebraic solutions, presented in [1]:

$$f_n^- = \frac{Det_{n-1}(a)}{Det_n(a)}\ ,\ f_n^0 = \tau + \frac{\tilde{D}_n(a)}{Det_n(a)}\ ,\ f_n^+ = \frac{Det_{n+1}(a)}{Det_n(a)}\ ,$$

where $Det_n(a)$ are the minors of order n of the following matrix:

$$\alpha = \begin{pmatrix} a[0] & a[1] & a[2] & ... \\ a[1] & a[2] & a[3] & ... \\ a[2] & a[3] & a[4] & ... \\ ...... & ...... & ...... & ... \\ ...... & ...... & ...... & ... \end{pmatrix}$$

Here $\tilde{D}_n(a)$ denotes that in the last row of the corresponding matrix the indices of $a[i]$ have been increased by one.

[1] *A.N.Leznov,M.A.Mukhtarov and W.J.Zakrzewski.* Tr.J.of Physics 19, 1995, 416
[2] *M.A. Mukhtarov.* Fizika 5, 2002, 38
[3] *M.A. Mukhtarov.* Proc.Ins.Math.Mech., 10(18), 2000, 123

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KİRAL SAHƏNİN MODELİNİN İNTEQRALLANMASINDA *MATHEMATICA* PROQRAMI

Əsas kiral sahəsinin məsələsinin inteqrallama prosedurun *Mathematica* alqoritmi təqdim olunub.

М.А. Мухтаров

ПРОГРАММА *MATHEMATICA* В ИНТЕГРИРОВАНИИ МОДЕЛИ КИРАЛЬНОГО ПОЛЯ

Представлен *Mathematica* алгоритм рекуррентной процедуры интегрирования задачи главного кирального поля.